Shortest scale of quantum field theory

Ram Brustein,* David Eichler,† Stefano Foffa,‡ and David H. Oaknin§
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel
(Received 8 September 2000; published 9 May 2002)

It is suggested that the Minkowski vacuum of quantum field theories of a large number of fields $\mathcal{N}$ would be gravitationally unstable due to strong vacuum energy fluctuations unless an $\mathcal{N}$ dependent sub-Planckian ultraviolet momentum cutoff is introduced. We estimate this implied cutoff using an effective quantum theory of massless fields that couple to semiclassical gravity and find it (assuming that the cosmological constant vanishes) to be bounded by $M_{\text{Planck}}/\mathcal{N}^{1/4}$. Our bound can be made consistent with entropy bounds and holography, but does not seem to be equivalent to either, and it relaxes but does not eliminate the implied bound on $\mathcal{N}$ inherent in entropy bounds.

DOI: 10.1103/PhysRevD.65.105013 PACS number(s): 11.10.Gh, 04.70.Dy

The existence of a fundamental value for the entropy of a black hole (BH) [1] which depends on a geometric property, its area, seems startling since it appears to limit the number of different types of elementary particles $\mathcal{N}$. If there were a sufficiently large number of particle species within a given mass scale, then any black hole much larger than that scale could have formed in a sufficiently large number of ways as to exceed any bound on entropy which does not depend on $\mathcal{N}$. This argument goes beyond holography [2,3], which states that the entropy is established at the boundary, but which nevertheless allows that surface-associated entropy to be proportional to $\mathcal{N}$. In fact, according to elegant arguments and calculations by Bombelli *et al.* and Srednicki [4], surface entanglement entropy is indeed proportional to $\mathcal{N}$ [5,6]. But why then should $\mathcal{N}$ be limited in any field theory that is to be consistent with gravity? Moreover, if gravity is the limit of a large N gauge theory [7] how could it disallow large $\mathcal{N}$?

In an attempt to broach these issues, we show here that three commonly made assumptions (i) that free, or weakly coupled quantum field theories (QFT's) have empty Minkowski space-time as their vacuum on scales greater than the familiar Planck length L_p, (ii) that free, or weakly coupled QFT's can be used to calculate the spectrum of quantum fluctuations in Minkowski space-time background and (iii) that quantum fluctuations gravitate as any other source of energy, are not compatible for sufficiently large $\mathcal{N}$. Assumptions (i),(ii) are commonly used to calculate many physical observables, and have been tested experimentally to very good accuracy at low energy scales. Assumption (iii) has not been tested experimentally, but is commonly believed to be correct, and leads to the infamous cosmological constant problem [8] due to the contribution of zero-point quantum fluctuations to the energy density.

We propose that the resolution to the conflict between assumptions (i)–(iii) when the number of fields becomes large is that quantum field theories which have flat space as their vacuum have an $\mathcal{N}$-dependent limit on their shortest scale (ultraviolet cutoff). The logical possibilities are that either $\mathcal{N}$ is limited, that the scale of quantum gravity can be substantially lower than Planckian, or that quantum fluctuations do not gravitate, making assumption (iii) invalid. The latter possibility would also be a possible solution to the cosmological constant problem.

We first show that the naive Minkowski vacuum of free QFT's of a large number of fields can be gravitationally unstable. For such theories, vacuum energy fluctuations in regions whose volume is smaller than a certain "critical" volume (which can, as large $\mathcal{N}$ is contemplated, be parametrically larger than a Planck volume) would become so strong that they induce them to collapse. The Minkowski vacuum then heads towards a BH slush unless some back reaction can modify the naive Minkowski vacuum sufficiently to prevent this final outcome. Since this can happen on scales parametrically lower than Planckian for large $\mathcal{N}$, the problem is not due to quantum gravity effects, rather, classical gravity is amplifying energy density seeds originating from field theoretic quantum fluctuations, in analogy to the germination of large scale structure during an inflationary phase of the early universe.

We are not suggesting that the vacuum actually is unstable. We merely point out that Minkowski space-time would be unstable if we try to extend field theory with a large number of fields and semiclassical gravity beyond a certain scale. As for what happens beyond that scale, perhaps the vacuum needs to be redefined by non-perturbative effects due to back-reaction on the vacuum (or the BH slush, should it come to that). In any case, the final outcome represents a significant modification of naive Minkowski space. While it may remain Minkowskian at sufficiently large scales, it features an ultraviolet cutoff that depends on $\mathcal{N}$, and thus restricts the number of independent degrees of freedom available in a region of given size. This number, when there is no cosmological constant, generally increases as fractional power of $\mathcal{N}$, so it implies neither an absolute entropy bound [9], nor a linear dependence of such a bound on $\mathcal{N}$, as in holography.

Energy fluctuations in the vacuum occur even though the vacuum is an eigenstate of the total Hamiltonian. For our concrete discussion of energy fluctuations in the vacuum we consider a free field theory of $\mathcal{N}$ massless bosonic scalar

*Electronic address: ramyb@bgumail.bgu.ac.il
†Electronic address: eichler@bgumail.bgu.ac.il
‡Electronic address: foffa@bgumail.bgu.ac.il
§Electronic address: doaknin@bgumail.bgu.ac.il

fields, but our results are applicable, with slight modifications, to physical components of any kind of fields, such as fermions, gauge bosons, etc. We assume that the fields' masses are protected from quantum corrections, for example, by supersymmetry, or a gauge symmetry, so they are strictly massless. The restriction to massless fields is mainly for convenience, allowing us to present simpler analytic results which capture the essence of our point. We further assume that the cosmological constant has been set to zero (at least to some accuracy), for example by unbroken supersymmetry. The curvature of spacetime is therefore much lower than Planckian, and we assume for simplicity that the background space is Minkowski space. We do expect, however, that the instability we will demonstrate persists in presence of a (small) cosmological constant. We neglect possible renormalization of Newton's constant, so that the scale of quantum gravity is indeed the Planck scale. Throughout we emphasize functional dependence on mass scales and $\mathcal{N}$ (which we assume to be a large parameter), and work in units in which $\hbar = c = 1$.

The Hamiltonian $H=\int d^3x\,\mathcal{H}(\vec{x})$, of a single massless scalar field ϕ in Minkowski spacetime is given by a volume integral over the Hamiltonian density $\mathcal{H}(\vec{x})=\frac{1}{2}[(\vec{\Pi}(\vec{x}))^2+(\vec{\nabla}\phi(\vec{x}))^2]$, where $\vec{\Pi}$ is the momentum conjugate to ϕ. Separating space into two parts, an "inside" region of volume V and an "outside" region of volume $\hat{V}$, the total Hamiltonian is simply given by $H=H_V+H_{\hat{V}}=\int_V d^3x\ \ \mathcal{H}(\vec{x})+\int_{\hat{V}} d^3x\ \ \mathcal{H}(\vec{x})$.

Although the vacuum state is an eigenstate of H, it is not an eigenstate of H_V or $H_{\hat{V}}$. So in spite of the vacuum being an eigenstate of the total Hamiltonian, the energy contained in the volume V is subject to fluctuations, its dispersion given by

$$\langle(\Delta H_V)^2\rangle=\frac{1}{8}\int_V d^3y_1 d^3y_2\left[\int\frac{d^3p\,d^3q}{(2\pi)^6}\left\{e^{-i(\vec{p}+\vec{q})\cdot(\vec{y}_1-\vec{y}_2)}\times\left[pq+2\vec{p}\cdot\vec{q}+\frac{(\vec{p}\cdot\vec{q})^2}{pq}\right]\right\}\right]. \tag{1}$$

Note that if V in Eq. (1) is the whole of space, then the integration over $\vec{y}_1$ and $\vec{y}_2$ produces a $\delta^3(\vec{p}+\vec{q})$, forcing the momentum integral to vanish. The dispersion of $H_{\hat{V}}$, as expected, is equal to that of H_V. This can be verified by expressing $\int_{\hat{V}}$ as $\int_{IR^3}-\int_V$ for the d^3x and d^3y integrations, and using the fact that each of the $\int_{IR^3}$ integrals gives a vanishing result due to the presence of $\delta^3(\vec{p}+\vec{q})$.

It is convenient to express Eq. (1) as an integral of a density, $\langle(\Delta H_V)^2\rangle=\int_V d^3y_1 d^3y_2 F(|\vec{y}_1-\vec{y}_2|)$, where the density of energy fluctuations $F(|\vec{y}_1-\vec{y}_2|)$ is given by the expression inside the square brackets on the right-hand side (RHS) of Eq. (1). Since F depends only on $x\equiv|\vec{y}_1-\vec{y}_2|$, we can perform all the integrals in Eq. (1), except for the x integral, by using the equality $1=\int_0^\infty dx\,\delta(|\vec{y}_1-\vec{y}_2|-x)$. The result is the following convolution:

$$\langle(\Delta H_V)^2\rangle=\int dx F(x)\mathcal{D}_V(x), \tag{2}$$

where the geometric factor $\mathcal{D}_V(x)$ depends only on the shape of the volume V.

The energy dispersion calculation leading to Eqs. (1),(2), has many similarities to the calculation of the entanglement entropy of a subsystem of a pure state [4]. Since the dispersion of H_V is equal to that of $H_{\hat{V}}$ they can depend only on properties of the common boundary of the two regions. This is perhaps counterintuitive; one might have expected the dispersion of H_V to be extensive, proportional to the volume V, but, as the previous argument shows, this is wrong. The fact that the dispersion of H_V has to be a function of boundary invariants and using dimensional analysis allows us to estimate it in different setups. Of course, as it stands, $\langle(\Delta H_V)^2\rangle$ is ultraviolet divergent, being an operator of mass dimension 2; to define it we have to introduce an ultraviolet momentum cutoff Λ. The exact form of implementing the cutoff will not affect the nature of our results, but it will change details, such as numerical coefficients of order of unity.

For the sake of concreteness and clarity, we restrict our attention for the moment to the case of a spherical volume V of radius R. We expect similar results when different geometries are considered, and present some examples later on. On dimensional grounds, the energy dispersion in a sphere of radius R, $\Delta E(\Lambda,R)=\sqrt{\langle(\Delta H_{Sphere})^2\rangle}$, is given by $\Delta E(\Lambda,R)=\mathcal{E}(R\Lambda)\Lambda$. We now proceed to find the analytical expression for the function $\mathcal{E}(R\Lambda)$. The geometric factor for a spherical volume is given by

$$\mathcal{D}_{Sphere}(x,R)=\frac{\pi^2}{3}x^2(x-2R)^2(x+4R), \quad 0<x<2R, \tag{3}$$

and, of course, vanishes for $x>2R$. Since the density F is ultraviolet divergent, it has to be regularized. We implement a particularly simple regularization procedure by inserting factors of $e^{-p/\Lambda}$ and $e^{-q/\Lambda}$ which suppress momenta larger than Λ in the momentum integrals of Eq. (1). Now we can explicitly evaluate F:

$$F(x,\Lambda)=\frac{\Lambda^8}{2\pi^4}\frac{3-10(\Lambda x)^2+3(\Lambda x)^4}{(1+(\Lambda x)^2)^6}. \tag{4}$$

Notice that F has an overall factor of Λ^8 as required by its dimensionality, that the maximal value of F is at zero $F(0,1)=3/2\pi^4\sim 0.015$, and that for large x, F is positive and decreases as x^{-8}. Using Eqs. (3) and (4), integral (2) for ΔE can be evaluated explicitly,

$$\Delta E(\Lambda,R)=\frac{(\Lambda R)^3}{\pi}\left[\frac{8[5+4(\Lambda R)^2]}{15[1+4(\Lambda R)^2]^3}\right]^{1/2}\Lambda. \tag{5}$$

For regions of different shapes and different cutoff procedures we expect similar results and indeed have found similar results.

In a theory of a large number of fields $\mathcal{N}$, the energy dispersion $(\Delta E_{\mathcal{N}}(\Lambda,R))^2=\langle(\Delta H_{Sphere})^2\rangle$ is proportional to $\mathcal{N}$, since the contribution of each field adds up linearly, so $\Delta E_{\mathcal{N}}(\Lambda,R)$ is given by

$$\Delta E_{\mathcal{N}}(\Lambda,R)=\sqrt{\mathcal{N}}\mathcal{E}(R\Lambda)\Lambda, \tag{6}$$

where $\mathcal{E}(R\Lambda)$ can be read off Eq. (5).

The energy fluctuation $\Delta E_{\mathcal{N}}(\Lambda,R)$ differs from the expectation value of vacuum energy $\langle H\rangle$ (the cosmological constant). For example, bosonic and fermionic fields contribute with different signs to $\langle H\rangle$, so an exact cancellation, as in a supersymmetric theory, is possible. But we have explicitly checked that bosonic and fermionic contributions to the dispersion have the same sign, as expected, and cancellation is not possible. In addition, their $\mathcal{N}$ dependence is different, $\Delta E_{\mathcal{N}}(\Lambda,R)$ being proportional to $\sqrt{\mathcal{N}}$, while $\langle H\rangle$ is generically proportional to $\mathcal{N}$. Moreover, since we are dealing with fluctuations, it is clear that $\Delta E_{\mathcal{N}}(\Lambda,R)$ should not be considered as ordinary, classical energy, but rather as a stochastic, fluctuating quantity, with a typical lifetime given by $T_f\sim\pi/\Lambda$. This estimate is based on the fact that the dominant contribution to the fluctuation is given by the high momentum modes, which have an energy of the order of the cutoff scale Λ. If we consider only modes with energy less than some maximal energy scale Λ^* ($\Lambda^*<\Lambda$), the energy fluctuation $\Delta E(\Lambda^*,R)$ which is a subdominant contribution to the total fluctuation $\Delta E(\Lambda,R)$ has a longer lifetime $T_f^*\sim\pi/\Lambda^*>T_f$.

With these remarks in mind, we now show that unless the number of degrees of freedom of QFT's is bounded their Minkowski vacuum is gravitationally unstable. Let us consider a theory of $\mathcal{N}$ massless scalar fields in a classical spacetime background. If spacetime curvature is smaller than Planckian, then according to assumption (iii), the energy-momentum tensor of the QFT can be consistently used as a source in the classical Einstein equations for the metric.

When the expectation value of the energy-momentum tensor vanishes (recall that we have assumed that the cosmological constant vanishes), one must also consider its fluctuations as a stochastic source in the Einstein equations (see, for example, [10]). Adopting this prescription, we consider the gravitational effects of the energy fluctuation in a given volume, Eq. (6), and we immediately encounter a potential problem. When a typical energy fluctuation is within its own Schwarzshild radius, $2G_N\Delta E_{\mathcal{N}}(\Lambda,R)\gtrsim R$, a BH could be created (G_N is Newton's constant).

That a fluctuation is within its own Schwarzschild radius is not sufficient information to determine whether a BH would actually be formed. An additional necessary condition is that the travel time of light through the collapsing region must be comparable to the mean lifetime of the energy fluctuation itself. This means that only energy fluctuations with a lifetime $T_f^*>R$ can create a black hole in a region of size R, so only modes with energy less than $\Lambda^*\sim\pi/R$ are relevant to this process, and therefore the previous condition for vacuum instability should be refined as follows:

$$2G_N\Delta E_{\mathcal{N}}(\Lambda^*,R)\gtrsim R, \tag{7}$$

which, together with Eq. (6), implies that the condition is

$$R^2\lesssim 2\pi\mathcal{E}(\pi)\sqrt{\mathcal{N}}L_p^2, \tag{8}$$

where $L_p=\sqrt{G_N}$ is the Planck length.

For a large enough $\mathcal{N}$, the size of created BH's is large in Planck units, so the initial induced curvature by each one of them is small.

The main limitation to our calculation comes from the fact that we have treated the BH's as classical objects; this is a consistent procedure as long as the evaporation time by emitting Hawking radiation $t_{\rm ev}=[5(8\pi)^4/4\pi^3](M^3G_N^2/\mathcal{N})$ is longer than the characteristic classical time scale $t_{\rm bh}=2G_NM$. By setting $2G_NM=R$, one finds that only BH's with $R>R_c=\sqrt{\mathcal{N}/640\pi}L_p$ can be treated classically. Thus the validity of the classical treatment requires that the RHS of Eq. (8) must be larger than R_c^2, and hence $\mathcal{N}\lesssim(1280)^2\pi^4\mathcal{E}^2(\pi)\sim0.5\times10^7$ [which by Eq. (8) requires $R\lesssim\sqrt{2560\pi^3}\mathcal{E}(\pi)L_p\sim50L_p$]. Since BH evaporation process is the inverse of BH formation process [which we have considered to derive Eq. (8)], their strength is determined by the same parameters and couplings. Therefore it is not possible to tune some of the parameters of the theory to avoid BH evaporation with the context of the argument. The relative strength of the BH formation and evaporation processes becomes a detailed quantitative question, which we cannot address using our methods beyond what we have just discussed. To determine more reliably the limitation imposed on our arguments by the BH evaporation process, a better treatment of quantum gravitational effects and their interplay with field theoretic effects is required.

If indeed BH's are created, they are created at a rate of about Λ^* and at a density of about close packing, making the vacuum of the theory very different than Minkowski spacetime, in contradiction with assumption (i).

The gravitational instability of flat space that we have noted can be avoided if Λ, the field theory UV cutoff, is low enough. Since $\Lambda^*\sim\pi/R<\Lambda$, we obtain the following bound on the ultraviolet cutoff of the theory:

$$\Lambda\lesssim\frac{\alpha}{\mathcal{N}^{1/4}}M_p, \tag{9}$$

where $M_p=1/L_p$ is the Planck mass, and $\alpha=\sqrt{\pi/2\mathcal{E}(\pi)}\sim2.9$. This bound is subject to the validity conditions which we have discussed above.

We have found that for a given Λ, treating $\mathcal{N}$ as a variable, there is a critical value $(M_p/\Lambda)^4$ above which the vacuum becomes gravitationally unstable. Alternatively, if $\mathcal{N}$ is fixed and we treat Λ as a variable, we find that Λ cannot be made larger than a certain critical value, which is parametrically lower than the Planck scale. Thus a large number of massless fields $\mathcal{N}\sim10^4$ (about the number of massless modes in some string theories) can be admitted in a field theory provided that the ultraviolet cutoff of the theory is sufficiently below the Planck scale. In the context of low energy effective field theories of weakly coupled strings, the cutoff scale is determined by the string length. However, according to our arguments, the magnitude of the cutoff scale may be set by considerations that do not seem to have a direct connection to the perturbative definition of the string length, rather by BH's that strings may form.

We conclude that in order to avoid a "granular collapse" of spacetime, the number of fundamental degrees of freedom has to be restricted if not bounded. Because our calculation assumes a flat background, we have not explicitly derived the mechanism by which the scale of QFT is cut off, nor proved that such a mechanism could be described by QFT. We suspect that the physics behind our argument is not unconnected to the constraints on the number of particles species that come from string theory or that appear to be implied by entropy considerations and holography. On the other hand, we have obtained our result without any reference to strings or entropy, and that raises the intriguing possibility that such implications of string theory may be more general than string theory itself.

The QFT cutoff Λ, one expects, somehow determines the area of a "single information bit" $A_{SIB} \sim 1/\Lambda^2$. We may now ask whether the size of A_{SIB} given by condition (9) is compatible with the proposed statistical explanation of BH entropy [1] as given entirely by entanglement entropy [4]. Recall that the entropy of a BH is proportional to its horizon area A_H in units of Newton's constant, $S_{BH} = A_H/4G_N$ and does not depend on $\mathcal{N}$, while entanglement entropy $S_{EN} = \mathcal{N}A/A_{SIB}$ depends linearly on $\mathcal{N}$. Considering S_{BH} and S_{EN} together in a way as to make them compatible without any bound on $\mathcal{N}$ would suggest that A_{SIB} should be proportional to $\mathcal{N}$. However, condition (9) suggests that the cutoff area $1/\Lambda^2$ scales only as $\mathcal{N}^{1/2}$. Thus, if the QFT cutoff determines the true size of a single information bit, we are left with an upper bound on $\mathcal{N}$. This upper bound, however, is not as strong as what one would obtain [11] by consideration of the entropy of a BH at the "naive" cutoff, namely the Planck scale, which admits $\mathcal{N}$ only somewhat greater than unity.

We acknowledge helpful conversations with R. Bousso and J. Friedman, and comments on the manuscript by J. Bekenstein. R.B. and S.F. are supported by the Israel Science foundation, S.F. is also supported by Della Riccia Foundation and by the Kreitman Foundation. D.E. is supported by the Israel Science Foundation, acknowledges the hospitality of the Institute of Theoretical Physics during completion of this paper, and the support of the National Science Foundation under Grant No. PHY94-07194.

[1] J.D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); **9**, 3292 (1974); S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[2] G. 't Hooft, in *Abdus Salam Festschrift: A Collection of Talks*, edited by A. Ali, J. Ellis and S. Randjbar-Daemi (World Scientific, Singapore, 1993), gr-qc/9310026; L. Susskind, J. Math. Phys. **36**, 6377 (1995).

[3] R. Bousso, J. High Energy Phys. **06**, 028 (1999); **07**, 004 (1999); Class. Quantum Grav. **17**, 997 (2000).

[4] L. Bombelli, R.K. Kuol, J. Lee, and L. Sorkin, Phys. Rev. D **34**, 373 (1986); M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).

[5] L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994).

[6] J.D. Bekenstein, gr-qc/9409015.

[7] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[8] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[9] J.D. Bekenstein, Phys. Rev. D **23**, 287 (1981); **49**, 1912 (1994); see also: Phys. Lett. B **481**, 339 (2000).

[10] See, for example, N.G. Phillips and B.L. Hu, Phys. Rev. D **62**, 084017 (2000); C. Kuo and L.H. Ford, *ibid.* **47**, 4510 (1993).

[11] J.D. Bekenstein, Gen. Relativ. Gravit. **14**, 355 (1982).